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Steady Flux in a Continuous-Space Sinai Chain

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We examine the steady-state flux of particles diffusing in a one-dimensional finite chain with Sinai-type disorder, i.e., the system in which in addition to the thermal noise, particles are subject to a stationary random δ-correlated in space Gaussian force. For this model we calculate the disorder average (over configurations of the random force) flux exactly for arbitrary values of system's parameters, such as chain length N, strength of the force, and temperature. We prove that within the limit $N \gg 1$ the average flux decreases with N as $\langle J(N) \rangle = C/\sqrt{N}$ and thus confirm our recent predictions that the flux in the discrete-space Sinai model is anomalous.

KEY WORDS: Fluctuation phenomena; transport in random media.

1. INTRODUCTION

Suppose an infinite line of integers and a particle which moves one step each unit oftime. Being at any chain's site j, $-\infty < j < \infty$, it moves to $j+1$ or $j-1$ with probability p_j or $q_j = 1 - p_j$, respectively. The set $\{p_j\}$ consists of independent identically distributed random variables, bounded away from 0 and 1 with expectation

$$E\left\{\log \frac{p_j}{1-p_j}\right\} = 0$$

and finite variance

$$0 < \sigma^2 = E\left\{\log \frac{p_j}{1-p_j}\right\}^2 < \infty$$

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This model of a random walk in a random environment was proposed by Sinai,[1] who has proved rigorously that the mean-square displacement $\langle X^2(t)\rangle$ of a particle for time t sufficiently large is proportional to $\log^4(t)$, in a striking contrast to the random walk in regular systems (all $p_j = 1/2$) with $\langle X^2(t)\rangle \propto t$.

The origin of this anomalous behavior can be readily understood on physical grounds (see, e.g., refs. 1 and 2). In one dimension it is always possible to define a potential energy function $U(j)$ so that the probabilities p_j satisfy the detailed balance condition,

$$\frac{p_j}{1 - p_{j+1}} = \exp[U(j) - U(j+1)]$$

The Sinai model represents a diffusive process in a potential $U(j)$. For a segment of size K this potential grows with K in proportion to $\sum_{j=1}^{K} \xi_j$, with ξ_j being independent random variables, $\xi_j = \ln[p_j/(1-p_j)]$. The fluctuations in $U(K)$ will grow like $\sqrt{K}$ for a typical environment $\{p_j\}$. Therefore, on each scale of size K one will have, typically, a potential barrier of height $\sqrt{K}$. The time required for a particle to diffuse over a barrier of size $\sqrt{K}$ is given by the Arrhenius law, $t_K \propto \exp(\sqrt{K})$. This suggests that on a given sample the displacement $X(t)$ of a particle in time t should grow like $\ln^2(t)$ for typical realizations of an environment $\{p_j\}$. Thus, one can expect $\langle X^2(t)\rangle \propto \log^4(t)$.

Following Sinai's work, this remarkable confinement of a random walk due to the random environment has attracted considerable attention. Different dynamical properties of this model, such as behavior of the diffusion front, the limit distribution, and the probability of presence at the initial point, have been discussed in great detail (see, e.g., refs. 2–5). In addition, this model was generalized[2,6] to include spatial correlations between the transition probabilities p_j. In ref. 6 these correlations were introduced directly into the statement of the problem, while in ref. 2 the correlations were induced by the particular algorithm—rarefactions of the Thue-Morse sequences,[2] which was invoked to constract the environment $\{p_j\}$. In both cases it was shown[2,6] that the logarithmic law for the second moment of particle displacement still holds, $\langle X^2(t)\rangle \propto \log^{\nu}(t)$, but the exponent ν is model dependent and might be different from the value 4. The behavior of the higher moments, however, might be effected dramatically due to spatial correlations in $\{p_j\}$.

In ref. 7 we examined some other aspects of the Sinai model. Consider a finite chain with some concentration of noninteracting particles, which execute random walk with Sinai hopping rules. Concentrations of particles at the endpoints, $P(X=0, t) = P_0$ and $P(X=N, t) = 0$, are kept fixed for all times. In such a system there exists a flux of particles to the trap at point $X=N$, which, for a fixed environment $\{p_i\}$, approaches at infinite time some constant sample-dependent value $J(N)$. This value is given by

$$J(N)=\frac{P_0}{2\tau(N)} \tag{1}$$

with

$$\tau(N)=\left[1+\frac{p_1}{q_1}+\frac{p_1 p_2}{q_1 q_2}+\cdots+\frac{p_1 p_2 \cdots p_{N-1}}{q_1 q_2 \cdots q_{N-1}}\right] \tag{2}$$

For the discrete-space model we could not take the average of $J(N)$ exactly even in the simplest case of a "two-state" distribution function

$$\rho(p_j)=\tfrac{1}{2}[\delta(p_j-p)+\delta(p_j+p)], \qquad \tfrac{1}{2}<p<1$$

However, analyzing the contributions of different realizations of an environment $\{p_j\}$ to the average flux, we were able to bound $\langle J(N)\rangle$ as

$$\frac{A}{N^{1/2}\ln^2(N)}\leqslant\langle J(N)\rangle\leqslant\frac{B}{N^{1/2}} \tag{3}$$

where A and B are some N-independent constants. Therefore, we were able to deduce[7] that the average steady flux behaves as $\langle J(N)\rangle \propto \phi(N)/\sqrt{N}$, where $\phi(N)$ is either a constant or decreases with N slower than any power of N. This result shows that the average flux behavior in the Sinai chain is anomalous. One can easily notice that $\langle J(N)\rangle$ is supported by atypical realizations of $\{p_j\}$. Flux stemming out of the typical realizations of $\{p_j\}$ behaves as $J_{\text{typ}}(N)=\exp\{\langle\ln[J(N)]\rangle\}\propto\exp(-\sqrt{N})$, i.e., the behavior of the disorder average flux $\langle J(N)\rangle$ differs markedly from the behavior of $J_{\text{typ}}(N)$.

Next, consider the flux of particles in the regular diffusive system in which all $p_j=1/2$. The Fickian formula for the flux in such a system follows from Eqs. (1) and (2), $J_F(N)\propto 1/N$. The comparison of Fickian flux with our prediction is quite surprising—in the present model with logarithmically confined random walk and exponentially large mean passage time one has the flux which is essentially greater than the flux in the regular system. In ref. 7 we analyzed the origin of such a behavior and have shown that it is supported by atypical bounded realizations of the random environment.

In the present communication we examine the behavior of the disorder average flux in the continuous-space analog of the original Sinai model. We evaluate the exact representation of $\langle J(N) \rangle$ which holds for an arbitrary chain length N. We prove that in the large-N limit the average flux follows the dependence $\langle J(N) \rangle = C/\sqrt{N}$, where the prefactor C is determined exactly.

2. BASIC EQUATIONS

The continuous-space analog of the Sinai model is provided by the Langevin equation (see, e.g., ref. 3)

$$\frac{dX}{dt} = \frac{1}{\gamma} F\{X\} + \eta(t) \tag{4}$$

where γ is a friction coefficient, η is the thermal noise,

$$\overline{\eta(t)} = 0, \qquad \overline{\eta(t)\,\eta(t')} = \frac{2T}{\gamma}\,\delta(t-t')$$

and $F\{X\}$ is a stationary random force which is a Gaussian white noise with

$$\langle F\{X\} \rangle = 0, \qquad \langle F\{X\}\,F\{X'\} \rangle = \Gamma\delta(X-X')$$

Here and henceforth a bar refers to the thermal averages for a fixed environment $F\{X\}$, whereas brackets stand for an average on the configurations of the random force.

The probability density $P(X, t)$ of the position X of the particle at time t satisfies, for a fixed environment $F\{X\}$, the following Fokker–Planck equation:

$$\frac{\partial P(X,t)}{\partial t} = D_0 \frac{\partial^2 P(X,t)}{\partial X^2} - \frac{1}{\gamma} \frac{\partial [P(X,t)\,F\{X\}]}{\partial X}$$

where D_0 is the diffusion constant in the absence of disorder, $D_0 = T/\gamma$, T being the temperature.

To calculate the steady-state flux we have to solve

$$\frac{d^2 P(X)}{dX^2} - \frac{1}{T} \frac{d[P(X)\,F\{X\}]}{dX} = 0 \tag{5}$$

subject to the boundary conditions

$$P(X=0) = P_0, \qquad P(X=N) = 0$$

Then the flux for a fixed environment $F\{X\}$ will be obtained from the equation

$$J(N) = -D_0 \left(\frac{d}{dX} - \frac{F\{X\}}{T} \right) P(X)$$

Solving Eq. (5), we evaluate the following explicit formula for the sample-dependent flux:

$$J(N) = \frac{D_0 P_0}{\tau(N)} \tag{6}$$

where

$$\tau(N) = \int_0^N d\xi \exp\left[-\frac{1}{T} \int_0^{\xi} d\xi' \, F(\xi') \right] \tag{7}$$

Equations (6) and (7) are simply the continuous-space versions of Eqs. (1) and (2).

In the remainder we will proceed as follows. First, we will calculate explicitly all the moments of the random function $\tau(N)$ in Eq. (7). Then we will restore the generating function,

$$\Phi(p, N) = \langle \exp[-p\tau(N)] \rangle$$

Eventually, the steady-state average flux will be obtained as

$$\langle J(N) \rangle = D_0 P_0 \left\langle \frac{1}{\tau(N)} \right\rangle = D_0 P_0 \int_0^{\infty} dp \, \Phi(p, N) \tag{8}$$

3. MOMENTS $\langle \tau^j(N) \rangle$

By definition

$$\langle \tau^j(N) \rangle = \int_0^N \cdots \int_0^N \prod_{k=1}^{j} d\xi_k \left\langle \exp\left[-\frac{1}{T} \sum_{k=1}^{j} \int_0^{\xi_k} d\xi'_k \, F(\xi'_k) \right] \right\rangle \tag{9}$$

To carry out the averaging of the exponent in Eq. (9) it is convenient to introduce a step function

$$\theta(\xi - \xi_k) = \begin{cases} 1 & \text{for} \quad \xi \leqslant \xi_k \\ 0 & \text{otherwise} \end{cases}$$

and rewrite Eq. (9) as follows:

$$\langle \tau^j(N) \rangle = \int_0^N \cdots \int_0^N \prod_{k=1}^{j} d\xi_k \left\langle \exp\left[-\frac{1}{T} \int_0^N d\xi \, F(\xi) \sum_{k=1}^{j} \theta(\xi - \xi_k) \right] \right\rangle$$

Averaging the exponent, we get

$$\langle \tau^j(N) \rangle = \int_0^N \cdots \int_0^N \prod_{k=1}^{j} d\xi_k \exp\left(\alpha \int_0^N d\xi \left[\sum_{k=1}^{j} \theta(\xi - \xi_k)\right]^2\right) \quad (10)$$

where we use the notation $\alpha = \Gamma/2T^2$.

Note that the integrand in Eq. (10) is the symmetric function of variables ξ_k. Therefore, choosing the sequence $N \geqslant \xi_1 \geqslant \xi_2 \geqslant \xi_3 \geqslant \cdots \geqslant \xi_N \geqslant 0$, we can express the j-fold integral in Eq. (10) as

$$\langle \tau^j(N) \rangle = \Gamma(j+1) \int_0^N d\xi_1 \int_0^{\xi_1} d\xi_2 \int_0^{\xi_2} d\xi_3 \cdots \int_0^{\xi_{j-1}} d\xi_j \times \exp\left[\alpha \sum_{k=1}^{j} (2k-1)\, \xi_k\right] \quad (11)$$

Performing the integrals in Eq. (11), we are led to the following recursive relations for the moments:

$$\alpha^j \langle \tau^j(N) \rangle = \frac{\Gamma(j)}{\Gamma(2j)} [\exp(\alpha N j^2) - 1] - \frac{1}{\Gamma(2j)} \sum_{m=1}^{j-1} \alpha^m \langle \tau^m(N) \rangle\, C_m^j \Gamma(j+m) \quad (12)$$

where C_m^j and $\Gamma(j)$ are the binomial coefficients and the gamma function, respectively. To calculate the moments $\langle \tau^j(N) \rangle$ explicitly, we note that Eqs. (12) have the formal solution

$$\langle \tau^j(N) \rangle = \alpha^{-j} \sum_{k=0}^{j} A_k(j) \exp(\alpha N k^2) \quad (13)$$

where $A_k(j)$ are N independent constants.

The coefficient $A_j(j)$ in Eq. (13) is obtained trivially—one can see that the leading large-N term in Eq. (13), $A_j(j) \exp(\alpha N j^2)$, is simply the first term in the rhs of Eq. (12), i.e.,

$$A_j(j) = \frac{\Gamma(j)}{\Gamma(2j)}$$

The second coefficient, $A_{j-1}(j)$, stems from the leading term of the previous moment, $\langle \tau^{j-1}(N) \rangle$. It is easy to see that

$$A_{j-1}(j) = -2j \frac{\Gamma(j)}{\Gamma(2j)}$$

The coefficient $A_{j-2}(j)$ is defined by the leading term of the moment $\langle\tau^{j-2}(N)\rangle$ and the second term of the moment $\langle\tau^{j-1}(N)\rangle$,

$$A_{j-2}(j) = -\frac{1}{\Gamma(2j)}\left[-C_{j-1}^{j}\Gamma(2j-1)A_{j-2}(j-1) + C_{j-2}^{j}\Gamma(2j-2)A_{j-2}(j-2)\right] = \frac{j(2j-1)\Gamma(j)}{\Gamma(2j)}$$

Evaluation of the remaining coefficients is a tractable combinatorial problem. Performing consequent calculations and taking into account the initial condition $\langle\tau^n(N=0)\rangle = 0$ for $n \geqslant 1$, we deduce the following formula for the moments of an arbitrary integer order:

$$\langle\tau^j(N)\rangle = \alpha^{-j}\frac{\Gamma(j)}{\Gamma(2j)}\left\{\sum_{m=0}^{j}(-1)^m C_m^{2j} \times \exp[\alpha N(j-m)^2] - \frac{1}{2}(-1)^j C_j^{2j}\right\} \tag{14}$$

One can easily verify this by the direct substitution of Eq. (14) into the recursive relations in Eq. (12). This is, in essence, the basic mathematical result of the present paper. It suffices to derive the generating function of $\langle\tau^j(N)\rangle$ and, eventually, the average steady flux.

4. GENERATING FUNCTION OF MOMENTS AND THE AVERAGE FLUX $\langle J(N)\rangle$

Making use of the identity

$$\exp(\alpha N j^2) = \frac{2}{(\pi\alpha N)^{1/2}}\int_0^\infty dx\, \exp\left(\frac{-x^2}{\alpha N}\right)\cosh(2xj)$$

we can cast Eq. (14) into the following compact form:

$$\langle\tau^j(N)\rangle = \frac{2\alpha^{-j}}{\Gamma(j+1/2)(\alpha N)^{1/2}}\int_0^\infty dx\, \exp\left(\frac{-x^2}{\alpha N}\right)\sinh^{2j}(x) \tag{15}$$

Consequently, one gets for the generating function

$$\Phi(p, N) = \sum_{j=0}^{\infty}\frac{(-p)^j}{\Gamma(j+1)}\langle\tau^j(N)\rangle = \frac{2}{(\pi\alpha N)^{1/2}}\int_0^\infty dx\, \exp\left(\frac{-x^2}{\alpha N}\right) \times \cos\left[2\left(\frac{p}{\alpha}\right)^{1/2}\sinh(x)\right] \tag{16}$$

The latter equation, however, cannot be applied directly for the calculation of $\langle J(N) \rangle$ in Eq. (8), since the orders of integration in Eqs. (8) and (16) cannot be interchanged. On the other hand, it is quite evident that $\Phi(p, N)$ is a decreasing function of the parameter p. It is easy to see that the generating function can be bounded from above as

$$\Phi(p, N) \leqslant \frac{2}{(\pi\alpha N)^{1/2}} \int_0^\infty dx \cos\left[2\left(\frac{p}{\alpha}\right)^{1/2} \sinh(x)\right]$$
$$= \frac{2}{(\pi\alpha N)^{1/2}} K_0\left(2\left(\frac{p}{\alpha}\right)^{1/2}\right)$$

where $K_0(z)$ is the McDonald function[8] of zeroth order. In the limit of large p one has explicitly

$$\Phi(p, N) \leqslant \frac{1}{(\alpha N)^{1/2}} \left(\frac{\alpha}{p}\right)^{1/4} \exp\left[-2\left(\frac{p}{\alpha}\right)^{1/2}\right]$$

i.e., $\Phi(p, N)$ drops off with p not slower than $\exp(-\sqrt{p})$.

To find a more plausible representation of $\Phi(p, N)$ than that in Eq. (16) we invoke the Kontorovich–Lebedev (KL) transform,[8] given by the pair of inversion formulas,

$$g(y) = \int_0^\infty f(x)\, K_{ix}(y)\, dx$$

$$f(x) = 2\pi^{-2} x \sinh(x) \int_0^\infty g(y)\, K_{ix}(y) \frac{dy}{y}$$

Here $K_{ix}(y)$ is the McDonald function of imaginary parameter, given, for instance, by

$$K_{ix}(y) = \int_0^\infty \exp[-y \cosh(t)] \cos(xt)\, dt$$

Let us define

$$g(y) = \Phi[y = 2(p/\alpha)^{1/2}, N]$$

and calculate $f(x)$ from the second KL formula. This yields

$$f(x) = \frac{2}{\pi} \cosh\left(\frac{\pi x}{2}\right) \exp\left(\frac{-\alpha N x^2}{4}\right)$$

Next, inserting the obtained $f(x)$ into the first KL formula, we obtain for the generating function

$$\Phi(p, N) = \frac{2}{\pi} \int_0^\infty dx \exp\left(\frac{-\alpha N x^2}{4}\right) \cosh\left(\frac{\pi x}{2}\right) K_{ix}\left(2\left(\frac{p}{\alpha}\right)^{1/2}\right) \quad (17)$$

We are now ready to evaluate the enclosed representation of $\langle J(N) \rangle$, which holds for an arbitrary value of N. Integrating Eq. (17) over p from 0 to ∞, one gets for the flux

$$\langle J(N) \rangle = \frac{2 D_0 P_0 \alpha}{\pi^2} \int_0^\infty dx \exp\left(\frac{-\alpha N x^2}{\pi^2}\right) x \coth(x) \quad (18)$$

The leading term in the large-N expansion of $\langle J(N) \rangle$ can be readily estimated. To do this, let us note that $x \coth(x)$ for arbitrary positive x can be bounded as

$$1 \leqslant x \coth(x) \leqslant 1 + x$$

Therefore, the following lower and upper bounds on $\langle J(N) \rangle$ are valid:

$$P_0 D_0 \left(\frac{\alpha}{\pi N}\right)^{1/2} \leqslant \langle J(N) \rangle \leqslant P_0 D_0 \left(\frac{\alpha}{\pi N}\right)^{1/2} \left[1 + \left(\frac{\pi}{\alpha N}\right)^{1/2}\right]$$

These bounds coincide in the limit $N \gg \pi/\alpha$ and therefore define the leading large-N term in $\langle J(N) \rangle$ exactly. It is also possible to write down an explicit large-N expansion of $\langle J(N) \rangle$ and thus define the correction terms. This is given by

$$\langle J(N) \rangle = D_0 P_0 \left(\frac{\alpha}{\pi N}\right)^{1/2} \left[1 + \sum_{n=1}^{\infty} \frac{(\pi^2/\alpha N)^n}{\Gamma(n+1)} B_{2n}\right] \quad (19)$$

where B_{2n} are the Bernoulli numbers.

Reintroducing the original dimensional parameters $D_0 = T/\gamma$ and $\alpha = \Gamma/2T^2$ into Eq. (19) we obtain for the average flux the following expansion:

$$\langle J(N) \rangle = \frac{P_0}{\gamma} \left(\frac{\Gamma}{2\pi N}\right)^{1/2} \left(1 + \frac{\pi^2 T^2}{3\Gamma N} - \frac{\pi^4 T^4}{15\Gamma^2 N^2} + \frac{2\pi^6 T^6}{63\Gamma^3 N^3} + \cdots\right) \quad (20)$$

5. CONCLUSIONS

To conclude, we have shown rigorously that the average steady-state flux of particles in a finite one-dimensional system with Sinai-type disorder is essentially non-Fickian. We have evaluated the exact formula for the average flux, which holds for arbitrary values of the system parameters, and have analyzed its asymptotic behavior. We have proved that for long chains, $N \gg 1$, the average flux is described by $\langle J(N) \rangle = C/\sqrt{N}$, where the prefactor C is determined exactly.

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